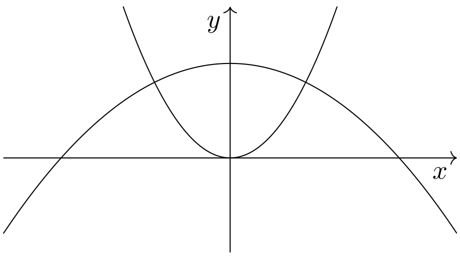
3001. A trapezium has lengths a, b, h, as in the standard area formula. Initially, $a = 2$, $b = 3$, $h = 4$, $\frac{da}{dt} = 1$, $\frac{db}{dt} = 2$, and $\frac{dh}{dt} = 6$. Find the initial rate of change of the area.

3002. A function f has instruction

$$\mathrm{f}(x) = \frac{2x+3}{x-1}.$$

Determine the indefinite integral of this function.

3003. The parabolae $y = x^2$ and $y = -\frac{1}{4}x^2 + k$, shown below, intersect at right angles.



Determine the value of the constant k.

3004. Three forces, whose magnitudes are $1, 2$ and 3 Newtons, act on an object. Their lines of action in the (x, y) plane are, not necessarily in order, $x = 1, 2, 3$. Show that the object cannot be in equilibrium.

3005. Show that, if $y = \cot 3x$, then $\dfrac{dy}{dx} + 3 + 3y^2 = 0$.

3006. Sketch $\sqrt{y} = \sin x$.

3007. On August 1st, a particular farmer has a 50% chance of starting the harvest. The next day, if he has not already started, he has a 52% chance of starting. The pattern continues with daily 2% increments in probability.

(a) Determine the first date by the end of which it is certain that the farmer will have started.

(b) Show that, up to this point, the probability p_n that the farmer will have started by the end of August nth is given by

$$p_n = 1 - \frac{25!}{50^n(25 - n!)}.$$

3008. Show that the lines $y = \left(2 \pm \sqrt{3}\right) x$ meet at 60°.

3009. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$\frac{y-b}{x-a} \geq \frac{1}{2}, \qquad (x-a)^2 + (y-b)^2 < 1.$$

3010. Two samples, with means $\bar{x}_1 \leq \bar{x}_2$ and standard deviations $s_1 \leq s_2$, are combined into a sample of mean $\bar{x}$ and standard deviation s. State, with a reason, whether the following claims are true:

(a) $\bar{x}_1 \leq \bar{x} \leq \bar{x}_2$,

(b) $s_1 \leq s \leq s_2$,

(c) If $\bar{x}_1 = \bar{x}_2$, then $s_1 \leq s \leq s_2$.

3011. Show that the equation below defines a rhombus, and sketch it, labelling the vertices.

$$\left|\frac{x}{a}\right| + \left|\frac{y}{b}\right| = 1.$$

3012. Function f has instruction

$$\mathrm{f}(x) = x + \frac{4}{x}.$$

This function is invertible when defined over the domain $[a, \infty)$ and codomain $[b, \infty)$. Determine the least possible values of a and b.

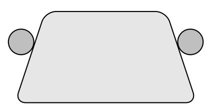
3013. Four points are defined as

$$\begin{aligned}
&A : \left(\sqrt{8/9}, 0, -1/3\right)\\
&B : \left(-\sqrt{2/9}, \sqrt{2/3}, -1/3\right)\\
&C : \left(-\sqrt{2/9}, -\sqrt{2/3}, -1/3\right)\\
&D : (0, 0, 1)
\end{aligned}$$

(a) Show that ABC is an equilateral triangle.

(b) Show that $ABCD$ is a regular tetrahedron.

3014. In a factory, a robotic claw picks up a machine part of weight 200 N. The claw clamps the part between two high-friction "fingers", with $\mu = 2$, as shown below in cross-section. The machine part may be modelled as an isosceles trapezium, with sides $4, 5, 4, 9$ cm.



(a) Draw a force diagram for the part, splitting contact forces into reaction and friction.

(b) Determine the minimum contact force for

i. equilibrium,

ii. an upwards acceleration of 0.5 ms^{-2}.

3015. Prove that the graph $y = x^3 - x + 2$ has rotational symmetry around the point $(0, 2)$.

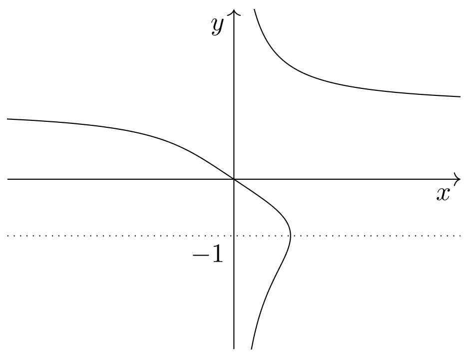
3016. A computer program assigns the function f as sin or cos with equal probability, and then chooses, at random, a value x from the domain $[0, \pi/4)$. Find $\mathbb{P}(\mathrm{f}(x) > 1/2)$.

FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP

3017. When moving through a fluid of high viscosity, the velocity of a particle, in ms^{-1}, may be modelled by $v = Ae^{-kt}$, for positive constants A, k. If such a particle has $v_0 = 6$ and $v_1 = 3$, determine the exact distance it travels during the first second.

3018. The diagram shows the curve $xy^2 - \dfrac{2x}{y} = k$.



Show that, whatever the value of the constant k, the curve crosses the line $y = -1$ at right angles.

3019. A function is defined, over a suitable domain, by

$$\text{f} : x \mapsto \frac{ax+b}{cx+d}.$$

Find a simplified expression for $\text{f}^{-1}(x)$.

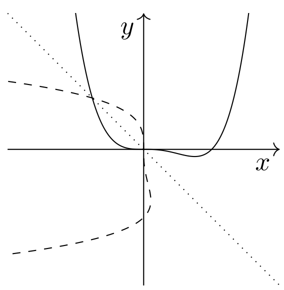
3020. For certain types of surface, the following fact is approximately true:

"A uniform cuboid, when placed on a rough slope, experiences the same frictional force irrespective of which face it stands on."

With reference to the coefficient of friction model and the definition pressure $= \frac{\text{force}}{\text{area}}$, explain why this happens.

3021. Show that $\displaystyle\int_0^1 \frac{2+\sqrt{x}}{2-\sqrt{x}}\,dx = 16\ln 2 - 9$.

3022. The diagram shows $y = \text{f}(x)$, and the image when it is reflected in the line $x + y = 0$.



Give the equation of the image.

3023. The numbers 1 to 6 are placed at random in the six sectors of a regular hexagon. Find the probability that the numbers alternate even/odd.

3024. It is given that, for $0 < a < b$,

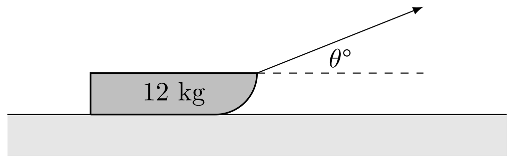
$$\int_a^b \frac{1}{x}\,dx = \Big[\ln x\Big]_a^b.$$

By considering $y = \frac{1}{x}$, prove that, for $a < b < 0$,

$$\int_a^b \frac{1}{x}\,dx = \Big[\ln(-x)\Big]_a^b.$$

3025. Show that (x, y) points on the curve $y = x^k$, where $k = 2 + \sqrt{3}$, satisfy $\log_x y + \log_y x = 4$.

3026. A child is dragging a 12 kg sledge across flat snow. The coefficient of friction is 0.125, and the string is angled at $\theta°$ above the horizontal. The tension in the string is 20 N.



The sledge is speeding up. Find the set of possible values of θ.

3027. This question is about the function

$$\text{f}(x) = \frac{1}{(2x-3)(3x-4)} + 25.$$

(a) Find the coordinates of the stationary point of $y = \text{f}(x)$. This point is a maximum.
(b) Give the equations of the one horizontal and two vertical asymptotes.
(c) Hence, sketch the curve.
(d) Explain why the Newton-Raphson method, with an arbitrary starting point x_0, would be unlikely to find either root of $\text{f}(x) = 0$.

3028. A polynomial graph of degree 10 has equation

$$y = \left(5x^2 - 6x - 1\right)^5.$$

Show that the three stationary points of the graph have x values which are in AP.

3029. If $\dfrac{d}{dx}\left(xy^2\right) = 1$, find $\dfrac{dy}{dx}$ in terms of x and y.

3030. A function h has instruction

$$\text{h}(x) = 2\sin x + \tan x - 2\cos x - 1.$$

The expression $\text{h}(x)$ may be written as the product of two factors, one linear in $\cos x$ and one linear in $\tan x$.

(a) Find these factors.
(b) Hence, solve $\text{h}(x) = 0$ for $x \in [0, 360°)$.

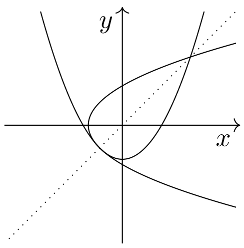
3031. Consider the two expressions $E_1 = 1 + x + x^2 + ...$ and $E_2 = (1 - x)^{-1}$, both defined over the same domain $|x| < 1$.

(a) Using a geometric sum, show that E_1 may be expressed as E_2.

(b) Using a binomial expansion, show that E_1 may be re-expressed as E_2.

3032. You are given that the parabolae $y = \frac{1}{2}x^2 + k$ and $x = \frac{1}{2}y^2 + k$ have four points of intersection. This question concerns the set of possible values of k.

(a) Explain why the following graph, in which there are two points of intersection, represents a boundary case:



(b) Hence, find the set of possible values of k.

3033. Two particles move with position vectors given by $\mathbf{r}_1 = t\mathbf{i} + 2t\mathbf{j}$ and $\mathbf{r}_2 = (3 - 2t)\mathbf{i} + (1 + t)\mathbf{j}$. Show that the particles collide.

3034. Prove that, if two parabolae $y = \text{f}(x)$ and $y = \text{g}(x)$ cross each other exactly once, then they must be translations of each other.

Note: the curves must actively cross, not merely intersect at a point of tangency.

3035. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a polynomial function f:

① if the polynomial $\text{f}(x)$ is divided by $(x - \alpha)$, the remainder is r,

② $\text{f}(\alpha) = r$.

3036. A straight line $y = k$ is drawn, which is tangent to the curve $y = x^6 - x^2$ at two points. Find, to 3sf, the area enclosed by the curve and this line.

3037. A function f is defined over the reals, and has range $(-a, a)$, where a is a positive constant. Give the range of each of the following:

(a) $x \mapsto \dfrac{1}{\text{f}(x) + a}$,

(b) $x \mapsto \dfrac{1}{2\,\text{f}(x) + a}$.

3038. The quintic approximation to $\sin\theta$ is

$$\sin\theta \approx \theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5.$$

Find the percentage error if this is used to evaluate

$$\int_0^{\frac{\pi}{6}} \sin\theta \, d\theta.$$

3039. Prove that, for any non-zero constants $a, b \in \mathbb{R}$, the simultaneous solution set of the following pair of equations contains precisely two (x, y) points:

$$ax + by = 0,$$
$$b^2x^2 + a^2y^2 = 1.$$

3040. Disprove the following statement:

$$\text{f}(x) \not\equiv \text{g}(x) \implies \text{fg}(x) \not\equiv \text{gf}(x).$$

3041. The following statement concerns a one-tailed test for a binomial probability p, with hypotheses

$$H_0 : p = 0.3$$
$$H_1 : p < 0.3.$$

"If k is an integer such that

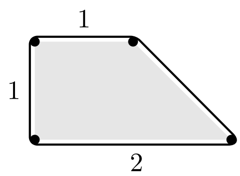
$$\mathbb{P}(X \le k - 1) < 0.05 < \mathbb{P}(X \le k),$$

then, at the 5% level, k is the critical value."

Identify the error and correct it.

3042. Show that the graph $y = \ln x + \ln(2 - x)$ has a stationary point on the x axis.

3043. In a machine, a light fan-belt rests in equilibrium around four small, smooth wheels, which sit at the corners of a right-angled trapezium. The tension in the fan-belt is T Newtons.



Find the magnitude of the force exerted on each of the four wheels by the fan belt, giving each force in the form $2T\cos\theta$.

3044. A function f is defined over the positive reals by

$$\text{f}(x) = x^2 - 3^{-x}.$$

(a) Use a sign-change method to show that the equation $\text{f}(x) = 0$ has a root.

(b) By differentiating, show that $\text{f}(x) = 0$ has no other roots.

3045. Three dice have been rolled, giving scores X, Y, Z. Determine whether knowing $X + Y = 8$ increases, decreases or doesn't change $\mathbb{P}(X + Z = 8)$.

3046. The equation of an ellipse is given as

$$x^2 + xy + y^2 = 3.$$

(a) Explain how you can tell that the ellipse is symmetrical in the line $y = x$.

(b) By finding the coordinates of the four points at which the tangent is parallel to either the x or y axis, sketch the ellipse.

3047. Find the largest possible real domain of

$$x \longmapsto \ln(x^2 - 1).$$

3048. A rightly concerned mouse of mass m has grabbed hold of the string of a rising balloon, and is now wondering when to let go. From rest at ground level, the balloon has exerted a steady upwards force of $\frac{3}{2}mg$ N on the mouse. Air resistance is taken to be zero.

(a) Show that, while holding on, at time t after lift-off, the height and velocity of the mouse are given by $h = \frac{1}{4}gt^2$ and $v = \frac{1}{2}gt$.

(b) The mouse lets go at $t = t_0$. Show that its landing speed is given by $v = kt_0$, where k is a constant to be determined.

3049. A function f is defined, over a suitable domain, by

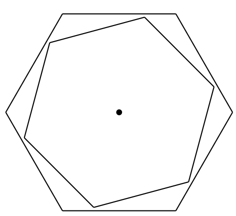
$$\mathrm{f}(x) = \sqrt{r^2 - x^2}.$$

A linear function g is then set up such that the following both hold:

$$\mathrm{f}\left(\tfrac{r}{2}\right) = \mathrm{g}\left(\tfrac{r}{2}\right),$$
$$\mathrm{f}'\left(\tfrac{r}{2}\right) = \mathrm{g}'\left(\tfrac{r}{2}\right).$$

Show that $\mathrm{g} : x \mapsto \frac{1}{\sqrt{3}}(2r - x)$.

3050. Find the area of the largest regular hexagon which can rotate freely while remaining inside a regular hexagon of area 4.

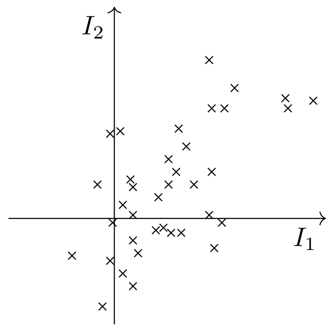


3051. Prove that the following equation has exactly one real root:

$$\frac{1}{(1+\sqrt{x})^3} + \frac{1}{(1-\sqrt{x})^3} = 1.$$

3052. Show that the parabolae $y = x^2$, $y = (x - k)^2$ and the x axis enclose a region whose area is $\frac{1}{12}k^3$.

3053. A statistics institute is examining the income of people who are self-employed. A sample is taken of net income I_1 and I_2 in two consecutive years, among those who were registered as self-employed for both years. The data are shown below, with profits positive and losses negative.



(a) Describe the relationship between I_1 and I_2, as shown by the sample data.

(b) Compare the number of people making losses in the first year and second year.

(c) Explain why such a sample is likely to give an unrepresentative picture of first-year income.

3054. By using a trigonometric substitution to integrate $y = \sqrt{1 - x^2}$ between suitable limits, show that the area of the unit circle is π.

3055. You are given that x and y are non-zero variables, and that the quantity $y^2(x + y)^{-1}$ is stationary with respect to x. Show that

$$\frac{dy}{dx} = \frac{y}{2x + y}.$$

3056. One of the following statements is true; the other is not. Identify and disprove the false statement.

① $\sin\phi = 0 \implies \sin 2\phi = 0$,
② $\sin\phi = 0 \impliedby \sin 2\phi = 0$.

3057. A variable y has a derivative which satisfies

$$\frac{dy}{dx} + 2y = 0.$$

(a) Verify that $y = e^{-2x}$ satisfies the above.

(b) A second solution is proposed, in the form $y = \mathrm{f}(x)e^{-2x}$ for some function f.

i. Find $\frac{dy}{dx}$ in terms of $e^{-2x}, \mathrm{f}(x)$ and $\mathrm{f}'(x)$.
ii. Show that $\mathrm{f}'(x) = 0$.
iii. Hence, prove that all solution curves must be of the form $y = Ae^{-2x}$, where A is a constant.

3058. Solve the equation $2\,\mathrm{cosec}\,t + \sin t = 3$, giving all values $t \in [0, 2\pi)$.

3059. It is given that f is an even polynomial function, i.e. f is a polynomial containing no odd powers of x. State, with a reason, whether it is certain that $y = \mathrm{f}(x)$ intersects the following curves:

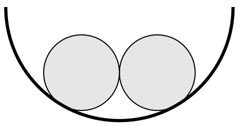
(a) $y = \mathrm{f}(-x)$,
(b) $y = -\mathrm{f}(x)$,
(c) $y = -\mathrm{f}(-x)$.

3060. Show that $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2}{1-x^2} + \frac{2}{4-x^2}\,dx = \ln 15$.

3061. In each case, decide which of the symbols $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ should occupy the space. Assume, in each case, that x and y are values for which the relevant functions are defined.

(a) $x \neq y$ $\qquad$ $\sin x \neq \sin y$,
(b) $x \neq y$ $\qquad$ $\arcsin x \neq \arcsin y$,
(c) $x \neq y$ $\qquad$ $|x| \neq |y|$.

3062. Inside a hemispherical bowl of radius $3r$, a pair of identical smooth spheres of radius r are resting in equilibrium. Show that the points of contact with the bowl are a distance $3r$ apart.



3063. Show that, for any $k > 0$, the following graph has three points of inflection which are collinear:

$$y = \frac{kx}{k + x^2}.$$

3064. A function f is defined over the reals by

$$\mathrm{f} : x \mapsto e^{2x}\cos^2 x + e^x \cos x.$$

Show that the range of f is $[-1/4, \infty)$.

3065. Assuming either of the standard versions of the chain rule, prove that the rate of change of $\mathrm{f}(x)$ with respect to $\mathrm{g}(x)$ is given by

$$\frac{d(\mathrm{f}(x))}{d(\mathrm{g}(x))} \equiv \frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}.$$

3066. Describe the transformation(s) that take the graph $y = (x-a)^2 + b$ onto the graph $y = k(x-a)^2 + b$.

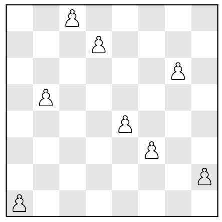
3067. Either prove or disprove the following statement: "If a quartic function f has $\mathrm{f}'''(p) = 0$ for some $p \in \mathbb{R}$, then $\mathrm{f}'(p) = 0$."

3068. Prove that $\displaystyle\int \tan x\,dx = \ln|\sec x| + c$.

3069. Variable P, which models profit, is defined by the equation $P = 6x + 9y$, where x and y are variables linked by $y = 3 - \frac{1}{2}x^2$.

(a) Explain why the profit will be maximised when the line $P = 6x + 9y$ is tangent to the curve $y = 3 - \frac{1}{2}x^2$.
(b) Find the maximal profit.

3070. Find the number of different ways of placing eight identical pawns on an 8×8 chessboard, such that no two pawns occupy the same row or column.



3071. Sketch the graph of $\mathrm{f}(x) = \operatorname{arccot} x$, specifying the domain and the range of the function.

3072. An equation is given as

$$\frac{\sqrt{x+1}}{\sqrt[3]{x-1}} = 1.$$

Show that this has no real roots.

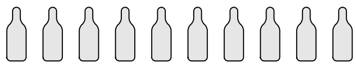
3073. Show that it is possible to fit a triangle with sides of length $7, 8, 9$ cm inside a rectangle of side length 6 cm.

3074. The monic cubic graph $y = \mathrm{g}(x)$ has a stationary point of inflection.

(a) Explain why g must be expressible in the form $\mathrm{g}(x) = (x-p)^3 + q$, for $p, q \in \mathbb{R}$.
(b) The point of inflection is at $(5, 6)$. Write down the values of p and q.

3075. Determine the area of a triangle whose vertices have position vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

3076. Ten green bottles are hanging on a wall, in a line.



Three green bottles accidentally fall, at random. Given that neither the first or last bottle falls, find the probability that the remaining seven bottles form exactly two distinct groups.

3077. Take $g = 10$ in this question.

A projectile is launched from the origin of an (x, y) plane, with a vertical velocity of 0.25 ms^{-1} and a horizontal velocity of u_x ms^{-1}. The equation of the trajectory is given by $5y = 8x - kx^2$. Find the value of u_x and the value of k.

3078. Show that $x+y=2$ is a tangent to the curve given implicitly by $xy(x+y)=2$.

3079. Prove that $\frac{d}{dx}(\cot x) \equiv -\operatorname{cosec}^2 x$.

3080. Three cards are dealt from a standard deck with the aces removed. Determine which, if either, of the following events has the greater probability:

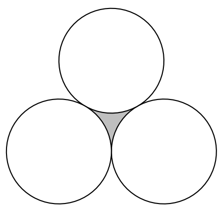
① three spades,
② three face cards (jacks, queens, kings).

3081. Sketch the following graphs on a single set of axes, clearly marking any intercepts.

(a) $y=x^6-1$,
(b) $y=x^8-1$.

3082. Solve $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}} = \dfrac{2+\frac{1}{x^2}}{1-\frac{1}{x^2}}$.

3083. Three circles, each of radius 1, are arranged such that each circle is tangent to both of the others.



Show that the area of the shaded region is $\sqrt{3}-\frac{\pi}{2}$.

3084. Show that $x=x^2+y^3$ has a line of symmetry.

3085. A old trick involves pulling a tablecloth out quickly from underneath the glasses on a dining table, without knocking them over. Explain, referring to Newton's second law and the coefficient of friction model, how the trick is possible.

3086. The equation of a curve is

$$e^y = \frac{x^2+3}{x-1}, \text{ for } x>1.$$

Find, in exact form, the coordinates of the turning point of the curve.

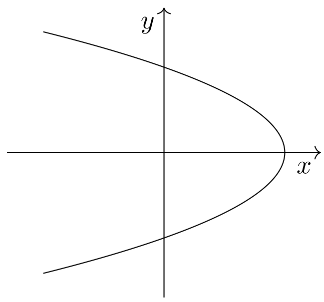
3087. Find the probability that three consecutive rolls of a die give scores s_1, s_2, s_3 in non-constant AP.

3088. Sketch $y=x^5-2x^4+x^3$.

3089. By finding an expansion up to x^4, show that, for small $|x|$, the function $\mathrm{h}(x)=(x+1)\cos\left(x^2\right)$ is very well approximated by a linear function.

3090. A sample of size n has $\sum x = 705$, $\sum x^2 = 2275$, and variance 1.1476. Find n.

3091. The diagram shows a curve defined by parametric equations $x=\cos 2t$, $y=\sin t$:



Find the Cartesian equation of the curve, giving x in terms of y.

3092. Equations in the variables x, y, z are given, in terms of the constants a, b, c, as

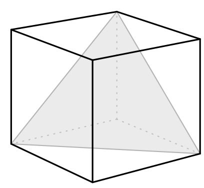
$$(x-a)(y-b)=0,$$
$$(y-b)(z-c)=0,$$
$$(z-c)(x-a)=0.$$

Solve these simultaneously for (x, y, z).

3093. Show that the line $2ay+x=2a^3+a$, where a is a constant, intersects $y=x^2$ at right angles.

3094. You are given that three forces, magnitudes $4, 5, 6$ Newtons, act on a 2 kg mass, which accelerates at 4 ms^{-2}. Show that this information is insufficient to calculate the angles between the lines of action of the forces.

3095. The diagram shows a cube of unit side length.



Three distinct vertices of the cube are chosen at random. Find the probability that the triangle so formed is congruent to the triangle depicted.

3096. The quadratic approximation to e^x, for small x, is given by $e^x \approx 1+x+\frac{1}{2}x^2$.

(a) Show that, at $x=0$, this approximation gives the correct values for the first and second derivatives of the exponential function.
(b) Show that this approximation does not give a correct value for the third derivative.

3097. A family of circles is defined, for $k \in \mathbb{R}$, by

$$(x-k)^2 + (y+k)^2 = 2.$$

The region R consists of the points of the (x, y) plane which lie on at least one of the circles. Sketch the region R.

3098. The following function is defined over D, the largest possible real domain:

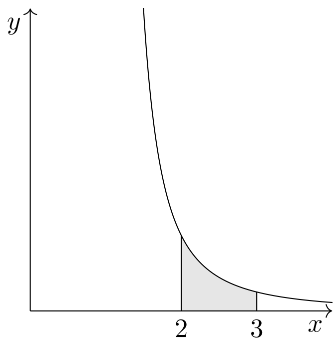
$$\mathrm{f} : x \mapsto \frac{\sin x}{1 - \sin x}.$$

(a) Find D.

(b) Find the range of the function.

3099. The diagram shows a region enclosed by the lines $x = 2$, $x = 3$, $y = 0$ and the parametric curve

$$x = 1 + \frac{1}{t},$$
$$y = t^2.$$



Determine the area of the shaded region.

3100. Sketch $y = x^3 - x^5$.

——— End of 31st Hundred ———